# Mathematical Modelling: A Pedagogic Context or a Boundary

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In this paper I will examine mathematical modelling in a school setting. Hence, it is presented as a pedagogic activity. I will contrast this activity with mathematical modelling in other professional settings and with mathematics in a school and professional setting. Mathematical modelling is commonly drawn on in the literature as a resource for supporting the learning of mathematics in schools, often as a motivating or contextualising mechanism. I will claim, however, that the activity of modelling is mythologised in these instances, since the recognition rules for the practice of modelling per se are not deployed. Instead, the principles of evaluation reside predominantly in the (school) mathematics deployed, although sometimes in perceived wider socio-political benefits. I will use as my data a school modelling activity developed for the Bowland Trust in England as a project led by Jeremy Burke and Jeremy Hodgen at King's College, London. The activity was trialled in a number of schools in the course of the project, but has been subsequently developed as a narrative focussing on the specific development of critical mathematical modelling. This narrative is detailed in Olley (2011).

In the professional world, mathematical modelling is a key tool used by engineers, managers, financiers and scientists alike. A brief review of papers describing mathematical models, suggests the level of complexity is far beyond the scope of a school course in mathematics. However, certain key elements can be engaged with, for example, the cyclical nature of the modelling process in which a model is suggested, critiqued and refined. We need to consider the relationship between the key indicators and the measures which are used to stand for them in the model. With respect to accuracy, the model is validated if the output it produces is sufficient for the specification. In modelling we are not interested in the correct answer, instead we are interested in a solution which is good enough. This last aspect is particularly alien to school mathematics.

In the educational setting mathematical modelling is presented in different ways. Most commonly it is presented in the contexts that authors of school materials present their examples and exercises in. Burke (in press) shows two such examples from leading English school teaching text books. For higher achieving students the path of a rugby ball being kicked over the posts is to be matched against graphs of its speed against time. Firstly, this is an example of what (Dowling 1997) refers to as the myth of reference. Clearly, the kicker cannot be credibly perceived as constructing the graph as a purposeful aid to the kick and yet none of the graphs given is a particularly reasonable representation of the change of speed and time of the ball. As Burke suggests; "the activity is reading a graph and the apparent non-mathematical context is a way in, a 'selling' point, for the problem. " (Burke, in press). We can confidently assert that this is not an example of mathematical modelling, in that it does not present any of the features suggested above, not least that outcome of the problem is never related back to the kicker, who actually has no interest in the solution. It is nonetheless presented here as the vast majority of examples in which mathematics appears to be applied in published teaching materials would be manifested in this form. The disconnect between problem and solution is the key indicator.

Sometimes the credibility of the problem leads mathematics educators to fall into the trap of believing the setting. Galbraith (Galbraith 2011) reports a question from the PISA project which provides a multiple choice set of answers to the question of how many fans could stand in a field at a concert 100m by 50m. The correct answer is given as C. 20,000. However, there is also an answer B. 5,000 which must be taken as incorrect. Galbraith says; “This sample item involves spatial insight, as students need to decide on a suitable model to quantify the amount of space occupied by a human, then perform an appropriate calculation to estimate how many people would fit into a given space. Only about 26% of the multinational sample of students answered the item correctly (C), illustrative of the depressed performance associated with contextualised problems.” It is worth considering why the authors of the question are sufficiently confident in their expertise in managing Rock concerts that they can assert one answer as correct and the other as incorrect. The field is 5000 square metres and the correct answer suggests that we could reasonably fit 4 people per square metre. Naturally, this would need to be an average, since it would be hard to conceive of every square metre being completely full especially around the edges, hence some of the square metres would need to fit rather more than 4 people. The UK Health and Safety Executive publishes it’s Purple Guide (HSE, 1999, p.17) which says; “Generally, 0.5 $m^2$ of available floor space per person is used for outdoor music events. “ What is interesting here is that the disconnect between question and answer is generated because the mathematics educator feels sufficiently confident in their own knowledge of settings of which they are clearly not expert. One critical feature of mathematical modelling is that the model is critiqued by the owners of the problem who will be expert in their field, and not the owners of the mathematics. (This problem was also raised in the US media (see Matthews 2009), but from a different perspective).

Dowling suggest a structuring of the domains of pedagogic action, this contrasts the public domain of a practice with its esoteric domain in terms of the strength of institutionalisation found in content and expression of the practice. (Dowling 1997, 2007 p.5). He represents this diagrammatically:

| | | Content | |
|---|---|---|---|
| | | I+ | I- |
| Expression | I+ | Esoteric domain | Descriptive domain |
| | I- | Expressive domain | Public domain |

The public domain is not general discourse, instead it is the discourse in the practice, characterised by low levels of institutionalised expression and content.

Realistic Mathematics Education is a movement in which the term model is used in a mathematical context. Indeed, “what is aimed for is a process of gradual growth in which formal mathematics comes to the fore as a natural extension of the student's experiential reality.” (Gravemeijer 1999) This is seen as a modelling activity, in which students construct increasingly formal (esoteric domain) mathematical statements developed from their public domain discourse in mathematical settings. “… RME models are not derived from the intended mathematics. These models are seen as student-generated ways of organising their mathematically grounded activity” (Gravemeijer et al. 2003). This certainly resonates with the professional practice of mathematical modelling, but the purpose remains rooted within (school) mathematics practice. Issues of validity are entirely within mathematics practice and directed at an induction into esoteric domain mathematics.

Cyril Julie (Julie 2002) set a group teachers a series of tasks rooted in issues of potential concern to them or their communities. Two contrasting examples were to find a model for pay scales in school, on the basis of equal pay for equal work. The second asked to model the accumulation of plastic shopping bags on school fences (a current environmental concern to that community). Julie notes that different teachers were more or less engaged with these tasks differentially according to the immediacy of the outcome to their professional position or political interest. It is clear here that where teachers engaged enthusiastically, the outcomes were potentially interesting models. The contrast I wish to make, is that the teachers were engaged in the practice of mathematical modelling. They used only the resources they already had to engage with issues such as critique and validity. Effectively, they looked for a formula which fitted well enough to appear workable. When promoted to improve a model for the human development index, teachers made minor changes to the formula rather than engage with what might count as an effective model. This contrasts with activity in what I will describe as (school) mathematical modelling practice, where the explicit intention is to highlight the practice as an induction into it, rather than deploy the practice itself.

We have identified four possible contexts in which we may site activity of this sort. They will be either located in a professional or pedagogic practice and concern themselves with either mathematical or mathematical modelling realisation principles. I wish to locate the pizza activity in the context of school mathematical modelling.

We have chosen the practice of school mathematical modelling. I have suggested above that this involves a critical engagement with the relationship between problem and model, together with issues of the relationship between measure and indicator and issues of accuracy and validity. In the esoteric domain, discourse in this practice would be rich in the engagement with this terminology. In this sense, although Julie's teachers are clearly engaging in a mathematical modelling discourse, they are doing this in a 'common sense' way, without the engaging in the esoteric discourse of modelling. In this analysis their discourse is public domain discourse in the domain of school mathematical modelling. That is not to say that the discourse is neither valid, nor useful. Indeed, it is clear that they have found models which they have found useful in their professional lives, beyond the pedagogic setting. Indeed the informal judgements of validity could be seen as esoteric domain discourse in a professional modelling practice, because here the realisation principle would include an effective solution, which this clearly is. However, in the pedagogic setting, the school modelling practice, the esoteric domain would require an elaboration of validity, not just the accepting of it. The question would be, how do we make the move from the public domain of school mathematical modelling, to enable the learner to engage in esoteric domain discourse? Fundamentally we need to pin down the practice that we are referring to.

In an earlier presentation (Olley, 2010) I have elaborated a structure proposed by myself and Jeremy Burke, which we describe as map, narrative, orientation (MNO). We suggest that pedagogic action proceeds from the teacher's conception of the practice. This conception constitutes their map of the practice. This may be entirely unarticulated and internal or, as we propose as a teacher development strategy, articulated and structured in detail. This can be seen in contrast to Shulman's pedagogic content knowledge (Shulman 1970) and later developments of this theory (Ball & others 2008).

Necessarily, map will be personal, rooted in the subject and pedagogical knowledge of the teacher, but not the closeness of fit to a presumed pre-existing definition of the practice. Without map, we cannot generate a narrative, a route through the practice, apprenticing the student into its esoteric domain.

Burke (in press) critiques different structures for school mathematical modelling for the lack of articulation of the relationship between the two contexts that the process must negotiate. The professional mathematical modeller is expert in modelling, but frequently not expert in the practice for which the model is sought. The contextualisation is the critical phase, but most notably the validity of the model can only be tested ultimately by the commissioner, whose expertise lies outside of the practice of modelling. Hence, our activity will seek to make this explicit and engage overtly with the location and basis of judgements of validity. So, our map structures school mathematical modelling as the cyclical creation, critique and refinement of a mathematical structure finding good enough measures for the key indicators which generates outcomes validated as sufficiently accurate with reference to the problem posed.

The initial phase of the narrative aims to imbue a sense of purpose. We place the problem in the context of the owner of a new pizza outlet, who wished to recruit a consultant to advise on the range of issues to consider in determining the profitability of the new enterprise. We, as mathematics educators are expert in neither running a pizza shop, nor business management, so it is important that we make clear our expertise. A more advanced version of the narrative would benefit from the skills of experts in either or both of these fields. So, we have a public domain discourse on pizza shop management. This always throws up one central issue (amongst many others): you can reach more customers if you can keep your pizza hot for longer. The students are surprisingly expert in pizza purchasing and hence are aware of the variety of packaging that pizza shops use to keep their delivery products fresh and the means by which they deliver them. Up to this point, we have been engaged in a marketing relationship. We have an activity to sell: mathematical modelling, and our marketing strategy is through the use of a compelling and apparently engaging setting. We have nonetheless mythologised pizza shop ownership; we have no means to credibly validate the outcomes.

Now our level of expertise increases, because now we are in a position to develop a model. Despite a lack of expertise in professional modelling, we have nonetheless engaged in the discourse in school mathematical modelling which has allowed us to construct a credible map. We need to find measures for the key indicators. 'Sufficiently fresh' is indicated by a minimum topping temperature of 48° (which we found initially with a 'taste' test in which a pizza cooled and was tasted until the taster considered it unacceptable). 'Reach more customers' is indicated by the time taken for the pizza to cool to 48° (given that we can find the average = maximum speed that the scooters delivering them travel at, and hence a circular route of that now calculable distance seems credible as a deliverable zone). In our work with students, different packaging types were tested, which added an interesting dimension to the model, but this does not impact the argument set out here. The relationship between measure and indicator is a very important site for critique, although at this stage we continued the first iteration of the modelling process with these face value acceptable measures. (One teacher improved on the circular zone, buy plotting routes using the calculated distance using Google Maps software, as different direction had different road and traffic conditions).

We now set up an experiment in which a pizza is heated, then allowed to cool plotting its cooling against time. Again we are clear to stress the limit of our expertise as mathematics educators in issues of experimental design. Collaborating with the science department who can focus on this aspect would be better. The narrative has reached the point where we need participants to reflect on the rate of change. Hence, a data sheet is given out which asks participants to say what they think the 'just cooked' temperature will be and give a reason. There is a space in which they will put the actual 'just cooked' temperature, when the experiment is started. As soon as the temperature probe has settled at the highest temperature, the timer starts and participants estimate the temperature after one minute, giving a reason, at the one minute point the temperature is announced and participants estimate for the end of the second minute and so on up to 10 minutes. As the experiment progresses, they are encouraged to refine their 'reason' and increasingly express it as a calculation, increasingly with more than one element. Finally, they are asked to estimate the long term temperatures (30 mins, 2 hrs, 24 hrs).

As the pizza cools, participants watch a graph of the cooling against time being generated by the data logging/streaming apparatus connected to the temperature probe (equipment details in appendix 1). Over the 10 minute period of the experiment, the cooling graph looks very linear indeed. Asked to describe the basis on which they estimated successive temperatures, the most common response is of the order of "roughly 2.4° per minute". Some participants say that the rate of decrease is changing from around 2.6° per minute to about 2.2° per minute. In our observations with the trial school classes, the normal class teacher took control of the dialogue and 'explained' how to find the linear function. The outcomes reported here come from the author managing the narrative with various audiences of maths and science teachers and teacher managers/educators. It is interesting to note how naturally a discussion of first and second differentials emerges in this setting. Participants were asked to hold the thought of the changing rate and see the effect of the initial model. This way of talking led to a natural set up of a model being the starting temperature minus the rate of decrease times the number of minutes, i.e. something like $92° - t\times2.4°$. This aspect of the narrative was also successfully played out in two of the schools, one selective, suburban, the other non-selective inner city. The issue of scaling is very neatly highlighted here. The modelling function is graphed on top of the data, using the software and starts in the correct place but has a massively too steep gradient. The graph is scaled in seconds and the 'error' is quickly spotted. When the new model $92° - (t/60)\times2.4°$ is graphed it provides an excellent fit. Participants are always happy to do a little 'tweaking' to improve the fit. It was never the case that teacher participants have protested that they know it cannot be linear at this stage. Now we are in a position to solve the pizza shop owner's problem. The pizza is still just acceptable at time $t$ where $92° - (t/60)\times2.4°=48°$.

We have now completed one iteration of the modelling process. Is the result good enough? The graph looks very linear over the range of the experiment and the solution is often only about one and a half times the experimental range. So, it is simple a judgement call. It quite probably is good enough for the pizza shop owner. Again, the owner is mythologised, we are simply keeping up the marketing ploy. However, the narrative has set up some unease. What about the fact that the rate seemed to change? We refer back to the estimates of long term temperatures. Using the functional model and values of $t$ for 30 mins, 2 hrs and 24 hrs, generate patently absurd values for the temperature. When the graph is rescaled with the maximum value of $t$ changed to these values the gradient appears absurdly steep. The context provides the critique. We know that pizzas, left to themselves do not freeze of their own accord. In class, students were provided with software to find alternative modelling functions to fit (a) the data and (b) the known behaviour of pizzas. Hence, the most natural solution to the rate of change slowing being a quadratic is quickly dismissed due to the failure of unattended pizzas to heat up again. This leaves two possible functions which meet both conditions being a reciprocal function and an exponential function. These provide extremely good fits to the data and meet both criteria and hence are validated in the context of the problem. Interestingly, the next iteration of critique demands validation beyond the scope of the mathematician. Why Newton's law of cooling is exponential requires an explanation rooted in chemistry and physics. Again, the mathematician must be clear about the limits of their expertise.

The best example of this project playing out in full with a class of students had the teacher return to the original problem, with the validated results from the linear model and as suggested earlier, map out the local area using Google Maps to show a distinctly non-linear target area for deliveries. A very satisfying result, albeit still from the 'sales' perspective. There never was a pizza shop, so we will never know.

The detailed presentation of the narrative is intended to support the case that the activity is deeply rooted in what would be recognised as mathematical modelling by a professional, inducting the participant into the esoteric domain of mathematical modelling, by strategically emphasising key elements of the modelling process. It is clearly a pedagogic activity. The narrative is structured to selectively expose the elements and the problem is structured and suppressed in strategic ways, because we are in fact not solving a real problem, which allows us to extend the remit of the original problem to see the effects of requiring a higher level of validation. Critically, I make no claims for this as a (school) mathematics activity. It may be the case that students have learned some maths in doing this, but that would need to be tested empirically. What I do wish to claim is that by strategically placing my activity as a (school) mathematical modelling practice, I need to reflect on my map of that practice. This gives the structure for the construction of a principled narrative, through which the participant can be inducted into the esoteric domain of this (school) mathematical modelling practice.

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